

SOLAR WIND MODELS BASED ON EXOSPHERIC THEORY

Inaugural-Dissertation zur Erlangung der Doktorwürde
der Naturwissenschaftlichen Fakultät der
Ludwig-Maximilians-Universität München

vorgelegt von Klaus Jockers

Oktober 1969

1970

1. Berichterstatter: Priv.-Doz. Dr. D. Pfirsch
 2. Berichterstatter: Prof. Dr. A. Schlüter
- Tag der mündlichen Prüfung: 14. 1. 1970



Solar Wind Models Based on Exospheric Theory*

K. JOCKERS

Max-Planck-Institut für Physik und Astrophysik, München

Received December 15, 1969, revised February 2, 1970

Due to the rare collisions exospheric theory should be well suited to describe the solar wind. Nevertheless, according to the work of Chamberlain it was thought that evaporation of particles cannot lead to the transonic flow present in the solar wind. Chamberlain assumed the radial electric field to be equal to that in a static atmosphere. However, the radial electric field has to be determined selfconsistently by the quasineutrality condition. Then one can construct transonic solutions, even if the particles are collisionless.

In the models the spherically symmetric outflow of an electron-proton plasma is considered in which the protons are collisionless. In one model the electrons are also assumed to be collisionless while in some others an isotropic electron temperature was explicitly prescribed. From comparison of the models with observation one can draw the following conclusions about the real solar wind:

1. while in the quiescent solar wind nonthermal heating of the protons may not be present it seems probable that wave particle interactions or other nonstationary effects lead to a fluidlike behaviour of the electrons even at quiet conditions.

2. the observed fact, that the protons have a lower velocity dispersion in the solar than in the antisolar direction is caused by a truncation of the distribution function because of the evaporation of the particles in the gravitational field.

Key words: solar wind — collisionless plasma

1. Introduction

It has been rather successful to treat the solar wind macroscopically as a single fluid, that is, with one temperature which is related to the pressure by an energy equation (Dessler, 1967). Nevertheless, there is some reason to discuss more accurately the stationary solar wind flow. As Hartle and Sturrock point out, the electrons and protons in the solar wind must be treated separately because of the small energy exchange between them. Thus we should adopt at least a two-component solar wind model. Moreover the mean free path of the particles is comparable with the macroscopic length scale (Lüst, 1962; Biermann, 1969; Hartle and Sturrock, 1968) so that the particle distribution of electrons and protons cannot be completely isotropic, as it is verified for the protons by the space probe observations (Hundhausen *et al.*, 1967). The exact microscopic treatment of a plasma component in which collisions are rare, but not completely absent, is very difficult. It is much easier, however, to treat the two limiting cases. The first case, when the collisions between particles of the

plasma component are frequent enough to preserve a maxwellian distribution, allows a hydrodynamic treatment; the second, when the collisions are completely absent, can be handled by exospheric theory using the collisionless Boltzmann or Vlasov equation. Hartle and Sturrock using spherical symmetry constructed a solar wind model with two hydrodynamic components. The present paper contains spherically symmetric models with either two collisionfree plasma components or collisionfree protons together with electrons for which an isotropic temperature is explicitly prescribed. In every case particles different from protons and electrons are neglected. By comparison between these, in some sense extreme, models, it should be possible to understand more deeply some characteristics of the real solar wind.

Density gradients in a plasma produce an electric field. If we assume a spherically symmetric outflow, in the solar wind this field is radially directed. In the hydrodynamic case, it can easily be eliminated from the basic equations, even if electrons and protons are considered separately. If the particles are collisionfree this is not the case, since the number of particles which exist at a given distance from the sun and also the relation between density and pressure depend on

* This work has been accepted by the Science Faculty of the Ludwig-Maximilians-Universität of Munich in fulfillment of Ph. D. requirements.

the shape of the combined potential of the electric and gravitational forces acting on the particles. In the following these potentials are denoted by F_e and F_p , where the index refers to electrons and protons respectively. Consider as an example some radially moving protons at a distance r_0 from the solar center. Which of these particles can reach or come from a distance $r_1 > r_0$, where $F_p(r_1) > F_p(r_0)$? For F_p monotonically increasing, protons having a total energy greater than $F_p(r_1)$ possess this property. However, if F_p has a maximum F_{pm} between r_0 and r_1 , the total energy of the protons must exceed F_{pm} . Protons with a lower energy, even if they exist at r_0 , need not be present at r_1 . Therefore, in the two cases, even if $F_p(r_0)$ and $F_p(r_1)$ remain unaltered, the particle distribution function and in consequence the particle density and the relation between pressure and density need not be the same.

Many authors who have investigated the problem of collisionless evaporation of charged particles from the atmosphere of the sun, e.g. Pikel'ner, 1950; Chamberlain, 1960; Aamont and Case, 1962; Jensen, 1963; Brandt and Cassinelli, 1966, assumed as electric field the so-called Pannekoek-Rosseland field, which is present in a static atmosphere of electrons and protons having the same temperature. This field decelerates the electrons and accelerates the protons in such a way that the combined gravitational and electric forces on electrons and protons and thus the potentials F_p and F_e become equal. Therefore, it leads to equal densities of electrons and protons; the flux of the electrons, however, as Pikel'ner points out, would exceed the proton flux by a factor of the squareroot of the mass ratio m_p/m_e (m_p, m_e are the masses of protons or electrons). Since this would cause a continuous charge deposition on the sun, the Pannekoek-Rosseland field cannot be realistic.

It is easy to estimate the true electric field of the solar wind and the true shape of the potentials F_p and F_e . Consider the macroscopic equations of a purely radial, steady and spherically symmetric two-component solar wind:

$$m_e u_e \frac{du_e}{dr} = -\frac{1}{n_e} \frac{d}{dr} (k T_{e||} n_e) - \frac{2}{r} k (T_{e||} - T_{e\perp}) - \frac{M_\odot m_e \gamma}{r^2} + e \frac{dV}{dr}, \quad (1.1a)$$

$$m_p u_p \frac{du_p}{dr} = -\frac{1}{n_p} \frac{d}{dr} (k T_{p||} n_p) - \frac{2}{r} k (T_{p||} - T_{p\perp}) - \frac{M_\odot m_p \gamma}{r^2} - e \frac{dV}{dr}, \quad (1.1b)$$

$$\frac{d}{dr} (u_e n_e r^2) = \frac{d}{dr} j_e = 0, \quad (1.2a)$$

$$\frac{d}{dr} (u_p n_p r^2) = \frac{d}{dr} j_p = 0. \quad (1.2b)$$

Here $u_{e,p}$ refers to the bulk velocity, $T_{e,p||}$ and $T_{e,p\perp}$ to the temperatures parallel and perpendicular to the radial direction, $n_{e,p}$ to the particle density and $j_{e,p}$ to the flux density of electrons and protons per steradian. The quantity r denotes the distance from the solar center, $M_\odot$ the solar mass, k the Boltzmann, γ the gravitational constant and e the charge of the proton. The quantity $-dV/dr$ is the electric field. As long as we assume infinite conductivity of the solar wind plasma, Eq. (1.1) is valid independent from the kinetic behaviour of the particles (Spitzer, 1962). As already mentioned, j_p must equal j_e . Moreover, since the Debye-length is very small compared with the macroscopic length scale, also the relation $n_e = n_p$ must be satisfied (Schlüter, 1950). This leads to $u_e = u_p$. Therefore, keeping this in mind in the following, the indices of j, n, u are omitted, when a misunderstanding is impossible. For the potentials F_p and F_e one can write:

$$F_p(r) = \frac{M_\odot m_p \gamma}{r_0} \left(1 - \frac{r_0}{r}\right) + e V(r),$$

$$F_e(r) = \frac{M_\odot m_e \gamma}{r_0} \left(1 - \frac{r_0}{r}\right) - e V(r). \quad (1.3)$$

At the reference level r_0 we say that $V(r)$ and therefore F_p and F_e vanish. In the case of the Pannekoek-Rosseland field, $V(r)$ has the form

$$V(r) = -\frac{m_p - m_e}{2e} \frac{M_\odot \gamma}{2r_0} \left(1 - \frac{r}{r_0}\right). \quad (1.4)$$

Near the sun where the bulk velocity and the temperature anisotropy are small and where T_e equals T_p , as we see from Eqs. (1.1) this field is indeed the true electric field. Thus in this range F_p is monotonically increasing. Since the electron mass is small, we can neglect the l.h.s. of Eq. (1.1a) also at great distances from the sun. Hence we get from Eq. (1.1a):

$$e \frac{dV}{dr} = \frac{1}{n_e} \frac{d}{dr} (k T_{e||} n_e) - \frac{2}{r} k (T_{e||} - T_{e\perp}). \quad (1.5)$$

According to Biermann the fact that there is only a slight difference between the averaged transport velocity of the sector boundaries of the interplanetary magnetic field from the solar surface to the earth and the flow velocity observed near the earth makes it likely that the streaming velocity is rather well constant over most of the sun earth distance; this means that $n\alpha r^{-2}$. Furthermore, the electron temperature is rather isotropic and very likely exceeds the proton

temperature which behaves about $\alpha r^{-1/2}$. From that it follows with the aid of Eq. (1.5) that the electric field falls off slower than $r^{-3/2}$. Thus the electric force, which accelerates the protons outwards, exceeds at great distances the decelerating gravitational force which diminishes αr^{-2} . In contrast to the monotonic shape of the potential F_p in the case of the Pannekoek-Rosseland field, the true potential F_p possesses a maximum and decreases at great distances from the sun. Accordingly the electrons are much more vigorously braked than in the case of the Pannekoek-Rosseland field. Whereas in the above mentioned papers the unrealistic Pannekoek-Rosseland field was a priori assumed, in the present work the electric field is selfconsistently determined. In the solutions, even in the completely collisionless case, the potential F_p has the expected single maximum.

The solutions are constructed in the following way. The collisionless theory is developed in Section 2. For well posed boundary conditions it allows to calculate the distribution function and its moments, especially the flux and the density, of the collisionfree protons as functions of the distance r and the electric potential V . Above we have seen that the distribution function at a distance r is not uniquely determined by the value of the potential at the same distance but is also influenced by the global behaviour of the potential. Hence the proton density depends not only on r and on the local electric potential $V(r)$ but also on the maximum value F_{pm} of the combined gravitational and electric potential and of the location r_m of this maximum. Since the total proton flux $4\pi j_p$ does not depend on the distance r , it is a function of F_{pm} and r_m only. If in the expression for the proton density we replace F_{pm} by j_p we get the proton density as function of the quantities $r, V(r), j_p, r_m$. Also the density of the collisionless electrons can be derived as a function of $r, V(r), j_e$ and the location of the maximum F_{em} of the potential F_e . In contrast to the protons, where r_m is a free parameter, in the case of the electrons we know a priori that, because of the monotonic behaviour of the potential F_e , the location of the maximum F_{em} must be at infinity. Thus we can forget about this dependence. If for the electrons an isotropic temperature is prescribed, we obtain by integration from Eqs. (1.1a) and (1.2a) a modified Bernoulli equation:

$$\frac{m_e}{2} j_e^2 \left(\frac{1}{n_e^2 r^4} - \frac{1}{n_e(r_0)^2 r_0^4} \right) \int_{r_0}^r \frac{1}{n_e} \frac{d}{dr} (k T_e n_e) dr + \frac{M_\odot m_e \gamma}{r_0} \left(1 - \frac{r_0}{r} \right) - e V(r) = 0. \quad (1.6)$$

From this equation the electron density can be obtained again as an implicit function of j_e, r and $V(r)$, provided the temperature of the electrons is constant or a function of the density only. Thus using the relation $j_e = j_p = j$, the quasineutrality condition for both the case of two collisionfree components as well as the case of macroscopically described electrons, takes the form

$$n_p(r, V(r), j, r_m) = n_e(r, V(r), j). \quad (1.7)$$

This is an equation for the function $V(r)$ and the numbers j and r_m . According to Eq. (1.3) $V(r_m)$ can be expressed by the quantities F_{pm} and r_m and thus by j and r_m . Therefore at $r = r_m$ Eq. (1.7) yields

$$n_p(r = r_m, j) = n_e(r = r_m, j). \quad (1.8)$$

If we eliminate with Eq. (1.8) the flux density j from Eq. (1.7) we arrive at an equation for the electric potential $V(r)$ which contains only the parameter r_m . Studying the solutions of this equation, one finds out that, as in the hydrodynamic one component models, there is only one critical value of r_m for which a continuous solution exists. This solution passes through a "critical" point $r_c > r_m$ which corresponds to the transonic point of the hydrodynamic models. Thus the quantities r_m, j and $V(r)$ can be determined uniquely.

For the macroscopically described electrons it seems desirable to prescribe the temperature not as a function of the density but rather as a function of position. It is clear that this is possible, but in this case we cannot get a simple equation analogous to Eq. (1.8) which allows to eliminate j . In order to avoid this difficulty at first under the assumption of isothermal electrons we calculated the flux density j and the distance r_m as described above. Since these quantities remain fixed when the electron temperature is changed in the region beyond the critical distance r_c , in a second step a new temperature profile in this region was prescribed as a function of the distance, and the final $V(r)$ was determined from Eq. (1.6) with $n_e = n_p(j, V(r), r, r_m)$. The result of the numerical calculations, which were carried out on the electronic computer G3 of the Max-Planck-Institute of Physics and Astrophysics, are presented and discussed in Section 3.

The paper is supplemented by three appendices, based on the theory of Section 2, giving more details about the construction of the solutions. For the reader who wants to get some insight into the basic properties of the problem without having to study the complex structure of the general equations, a

very simple solar wind model is presented with isothermal electrons and collisionless radially moving protons in Appendix 1. Even though this model can be treated completely with analytical and graphical methods, it shows all reliant features of the general, more complicated problem. Appendix 2 contains a short derivation of the explicit expressions for the particle density, the flux density, and the temperatures of protons and electrons as functions of the potential F_p and F_e , respectively. In Appendix 3 some problems concerning the selfconsistent determination of the potential $V(r)$ are discussed.

2. The Collisionfree Plasma Component

In this section we consider the electric field and therefore the potentials F_p and F_e as known functions and construct the distribution function of collisionfree electrons and protons. In addition to the already mentioned gravitational and electric field we allow for a magnetic monopole field, which is the only magnetic field compatible with a spherically symmetric solar wind. In order to satisfy Gauss' theorem, we assume that the magnetic field has a sector-like structure, so that the total magnetic flux becomes zero even if the field within the sectors behaves like a monopole field. As in each spherical solar wind theory, the contribution from the small sector boundaries must be neglected. Within the sectors, this magnetic field does not influence the flow.

When we deal with the Vlasov equation it is convenient to use hamiltonian coordinates. In our case the hamiltonian reads

$$\mathcal{H} = \frac{1}{2m} \left[p_r^2 + \frac{p_\varphi^2}{r^2} + \left(\frac{p_\vartheta + e B_0 r_0^2 \sin \varphi}{r \cos \varphi} \right)^2 \right] + F(r), \quad (2.1)$$

where the quantity B_0 denotes the absolute value of the radial magnetic field at the reference level r_0 . Since Eq. (2.1) applies for electrons and protons in the same way, the indices of m and F have been dropped. The quantities r, φ, ϑ are polar coordinates with respect to the solar center; the angle φ refers to the heliographic latitude and the angle ϑ to the heliographic longitude. If we use the relation between the rectangular components $v_r, v_\varphi, v_\vartheta$ of the velocity in the r, φ, ϑ direction and the time derivatives $\dot{r}, \dot{\varphi}, \dot{\vartheta}$

$$\dot{r} = v_r, \quad r\dot{\varphi} = v_\varphi, \quad r\dot{\vartheta} \cos \varphi = v_\vartheta, \quad (2.2)$$

we get from the canonical equations the meaning of the canonical momenta $p_r, p_\varphi, p_\vartheta$:

$$p_r = m v_r, \quad p_\varphi = m r v_\varphi, \quad p_\vartheta = m r \cos \varphi \cdot v_\vartheta - e B_0 r_0^2 \sin \varphi. \quad (2.3)$$

In the canonical variables the stationary Vlasov equation has the following simple form:

$$\frac{\partial f}{\partial r} \frac{\partial \mathcal{H}}{\partial p_r} + \frac{\partial f}{\partial \varphi} \frac{\partial \mathcal{H}}{\partial p_\varphi} + \frac{\partial f}{\partial \vartheta} \frac{\partial \mathcal{H}}{\partial p_\vartheta} - \frac{\partial f}{\partial p_r} \frac{\partial \mathcal{H}}{\partial r} - \frac{\partial f}{\partial p_\varphi} \frac{\partial \mathcal{H}}{\partial \varphi} - \frac{\partial f}{\partial p_\vartheta} \frac{\partial \mathcal{H}}{\partial \vartheta} = 0. \quad (2.4)$$

This equation states that the distribution function f is constant along phase space trajectories. The phase space trajectories are characterized by constants of motion. Since we deal with a stationary and spherically symmetric problem there are many phase space trajectories which can be transformed into each other simply by time translation or rotation around the center of symmetry. All the phase space trajectories in such a group have the same energy and the same absolute value of the angular momentum and obviously they can be treated in the same way. Thus in our case the energy and the absolute value of the angular momentum are important constants of the motion almost sufficient to characterize the trajectories. In canonical coordinates the energy E is given by the hamiltonian. Instead of the absolute value of the angular momentum we use

$$I = \frac{1}{2m} \left[p_\varphi^2 + \left(\frac{p_\vartheta + e B_0 r_0^2 \sin \varphi}{\cos \varphi} \right)^2 \right]. \quad (2.5)$$

As can easily be seen from (2.3), $2mI$ denotes the square of the angular momentum.

It is clear, that each function of E and I , if we express these constants of the motion by the canonical variables, is a solution of the Vlasov equation. Nevertheless the totality of one-valued functions of E and I does not represent all spherically symmetric solutions of the stationary Vlasov equation. In order to explain that, we consider the radial motion of the particles. From Eqs. (2.1) and (2.5) we get

$$\dot{r} = p_r = \sqrt{2m \left(E - \frac{I}{r^2} - F(r) \right)}. \quad (2.6)$$

Since p_r must be real, for given values of E and I there exist orbits only in intervals where $E - I/r^2 - F(r) \geq 0$. Thus, in general, there are values of E and I for which exist particle orbits in some separate intervals of r . Clearly there is no contradiction to the Vlasov equation, if the distribution for these values of E and I in the separate intervals is different. Moreover, a distribution function which is a one-valued function of E and I predicts the same probability for incoming and for outgoing particles. On orbits, however, with $E - I/r^2 - F(r) > 0$ in the whole interesting interval, say from the reference

level r_0 until infinity, it is allowed to prescribe different values of the distribution function for the different signs of p_r since the outgoing trajectories are not connected with the incoming ones.

The above discussion will become more obvious, if we consider Fig. 1. For a given potential F as shown in the upper part of the figure the projection of the surface $E = I/r^2 - F(r)$ on the $\{E, I/r_0^2\}$ -plane is depicted. Real trajectories exist only above this surface. We want to prescribe the distribution function at $r = r_0$ and look for a solution of the Vlasov equation for $r > r_0$. Since at r_0 the inequality $E > I/r_0^2$ holds (remember that $F(r_0) = 0$), only the region above the diagonal $E = I/r_0^2$ is influenced, if we prescribe the distribution function at r_0 . This region can be divided into two parts; the upper part, which lies above all lines $E - I/r^2 - F(r) = 0$, corresponds to the orbits, which exist from r_0 until infinity. There the distribution function at r_0 can be presented as a function of E, I and the sign of p_r . By this choice the distribution function in this part of the $\{E, I/r_0^2\}$ -plane is determined uniquely for all distances $r > r_0$. In the other part, which is the remainder of the region $E > I/r_0^2$, there are two types of orbits: the first type belongs to particles, which move outward from r_0 , turn around somewhere between r_0 and r_m and fall again back to the sun. Since, because of the reflection, incoming and outgoing particles in this region are equally probable, the distribution function at r_0 in this region must not depend on the sign of p_r . The second type corresponds to particles which come from infinity, are reflected somewhere between r_m and infinity and fly back to infinity. These orbits are independent from the distribution function at r_0 . In order to determine the solution of the Vlasov equation uniquely, for these orbits another function of E and I must be prescribed. In the same way also for the region below the diagonal $E = I/r_0^2$, if there are real trajectories, the type of the trajectories must be determined and the number of particles on these trajectories must be prescribed as a function of E and I .

What kind of boundary condition must we apply at r_0 ? In reality, since the density and temperature of the plasma decrease monotonically outward, with increasing distance from the sun, the flow must gradually change from a collision-dominated to a collisionfree state. Following most earlier work (Jeans, 1954; Chamberlain, 1960) we idealize this gradual transition by an abrupt unsteady change and assume that up to this discontinuity, which we now identify with the above introduced reference layer

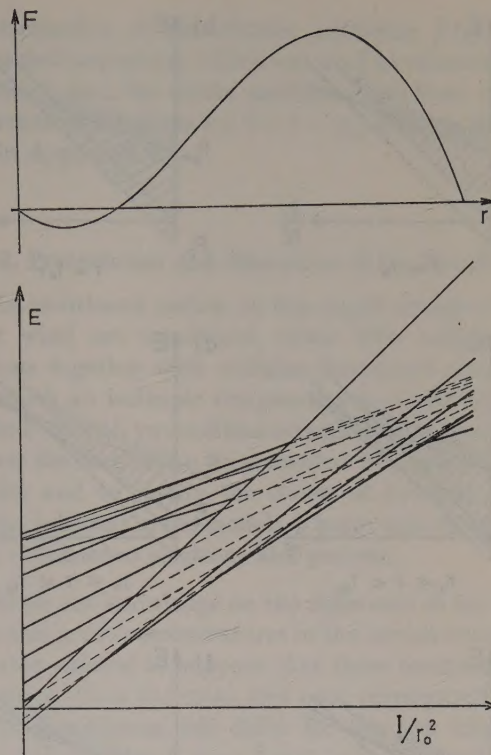


Fig. 1. The surface $E = I/r^2 + F(r)$ in $\{E, I/r_0^2, r\}$ -space is represented for a potential $F(r)$ given in the upper part of the figure. For different values of r the straight lines $E = I/r^2 + F(r)$ are plotted in the $\{E, I/r_0^2\}$ -plane. If one supplements this plane into the three-dimensional $\{E, I/r_0^2, r\}$ -space by an r -axis pointing away from the observer then the light and broken lines represent the hidden portion of the surface

located at r_0 , the atmosphere is still static. Since there are no particles coming from infinity, the orbits of such particles can only be occupied by collisions. Thus the particle distribution, which corresponds to a static maxwellian distribution just below r_0 , becomes abruptly depleted for particles incoming from infinity when we pass through the transition layer; on the other hand the distribution is continuous at r_0 for bounded and for outgoing particles. Thus we arrive at the following boundary condition for the distribution function. On escaping orbits, i.e. for $E - I/r^2 - F(r) > 0$ in the whole interval $r_0 \leq r < \infty$, the distribution function at r_0 has the form

$$f(r_0) = n_0 (2\pi k T_0 m)^{-3/2} e^{-E/kT_0} \frac{1 + \text{sign } p_r}{2}, \quad (2.7a)$$

while otherwise the term involving p_r must be omitted:

$$f(r_0) = n_0 (2\pi k T_0 m)^{-3/2} e^{-E/kT_0}. \quad (2.7b)$$

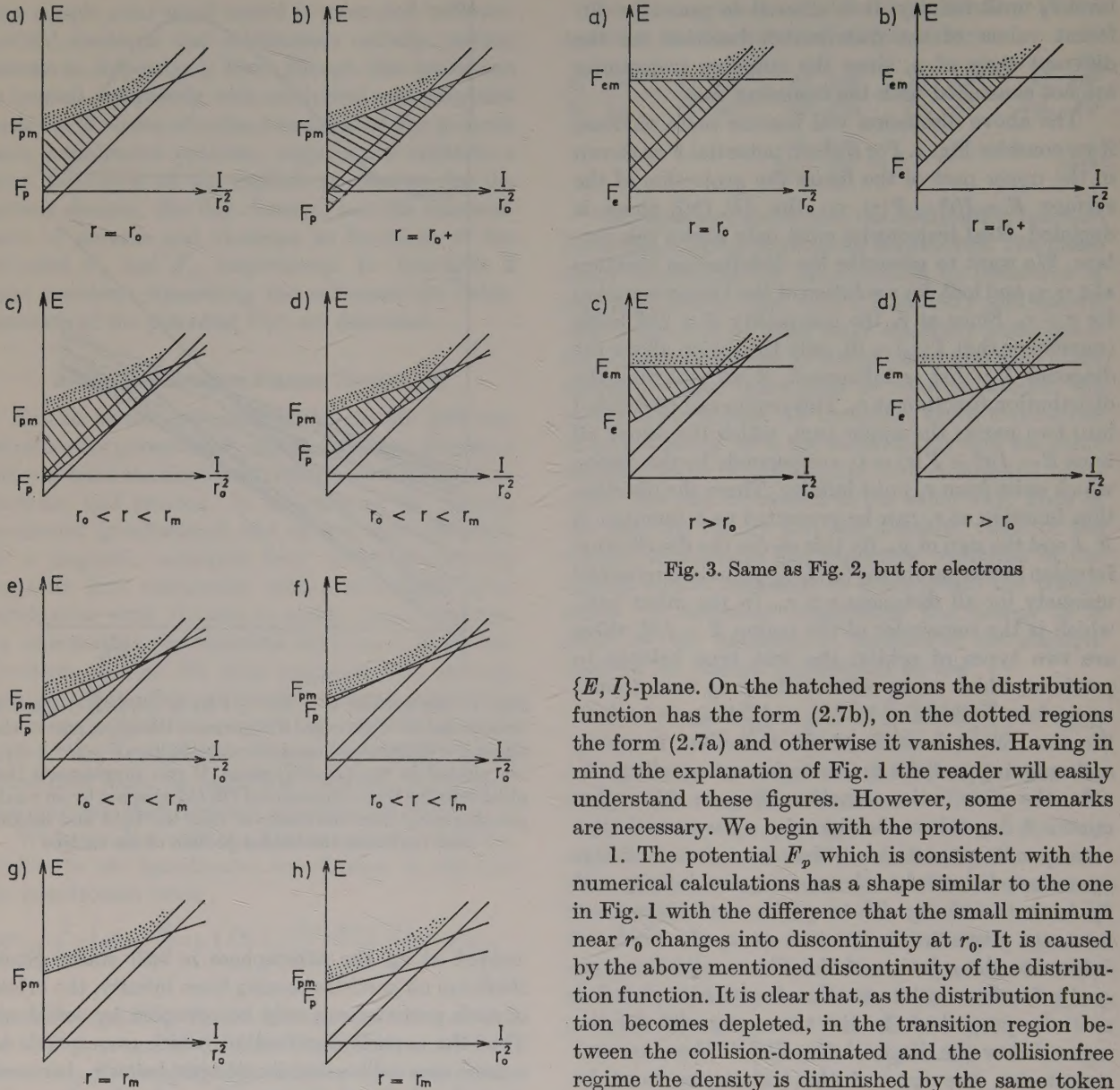


Fig. 2. Several cuts through the $\{E, I/r_0^2, r\}$ -space perpendicular to the r -axis. On these cuts the behaviour of the proton distribution function f_p is sketched. Dotted region: f_p as in Eq. (2.7a). Hatched region: f_p as in Eq. (2.7b). Otherwise $f_p = 0$. $r_0 +$ denotes a distance a little bit greater than r_0

T_0 denotes the temperature of the maxwellian distribution and n_e the corresponding density.

Figures 2 and 3 illustrate the distribution function of the collisionless protons and electrons for $r \geq r_0$, as they were used in the numerical calculations. One sees cuts through the $\{E, I, r\}$ -space parallel to the

Fig. 3. Same as Fig. 2, but for electrons

$\{E, I\}$ -plane. On the hatched regions the distribution function has the form (2.7b), on the dotted regions the form (2.7a) and otherwise it vanishes. Having in mind the explanation of Fig. 1 the reader will easily understand these figures. However, some remarks are necessary. We begin with the protons.

1. The potential F_p which is consistent with the numerical calculations has a shape similar to the one in Fig. 1 with the difference that the small minimum near r_0 changes into discontinuity at r_0 . It is caused by the above mentioned discontinuity of the distribution function. It is clear that, as the distribution function becomes depleted, in the transition region between the collision-dominated and the collisionfree regime the density is diminished by the same token and therefore the electric potential must adjust itself in a new way to preserve neutrality. This leads in the idealized picture of an infinitesimally thin transition layer to a discontinuity of the electric potential. For definiteness in the calculations it has been assumed that this discontinuity lies completely in the collision-free region. Since this is actually not the case, the height of the jump in the potential gives us a measure for an error unavoidably inherent in the model.

2. As we see from Figs. 2a–e, in the interval $r_0 < r < r_m$ an equilibrium distribution was prescribed on most of those orbits on which the distribution function is not already determined by the boundary

condition. We assume that such a distribution will be generated by rare collisions. This choice does not influence the distribution for $r > r_m$ and is therefore less important.

3. As indicated in Fig. 2a, all particles, which can exist at r_0 and r_m , are considered as escaping. Fig. 2f, however, illustrates that some of these particles are reflected at distances near r_m . Since the exact treatment would complicate the calculations very much, the mentioned simplification was introduced. The resulting error in the particle flux is in general less than 1%.

4. At $r > r_m$, there should exist only particles on escaping orbits as is indicated in Figs. 2g, h. It is interesting to note that, if only escaping particles were present, it would be impossible to determine a selfconsistent potential F_p in a small interval just beyond the distance r_m . However, due to rare collisions there always will be some particles in the region $E < I/r_m^2 - F_{pm}$. It is shown in Appendix 3 that there are many possibilities to prescribe distribution functions for these particles which allow for selfconsistent continuous solutions in the whole interval $r_0 \leq r < \infty$, even if the total number of these particles is very small. In the calculations presented below, for $r > r_m$, in the region $E < I/r_m^2 - F_{pm}$ but $E > I/(r_m \cdot 1.05)^2 - F_{pm}$ the distribution function of the protons was assumed equal to (2.7b) multiplied with a factor 0.05.

For the electrons the situation is less complicated because of the monotonic potential. Clearly the jump in the potential at r_0 is present also in the case of the electrons. In analogy to the protons it was assumed that all those particles are escaping which according to their $\{E, I\}$ -values can exist at r_0 and infinity. Because of the jump of the potential this leads to a small error similar to that discussed in point 3 above. For the bounded electrons, which do not reach r_0 , the equilibrium distribution was prescribed.

Having now specified the distribution function of protons and electrons, it is in principle easy to obtain the expressions for the particle density, flux density, velocity, and the temperatures $T_{||}$ and $T_{\perp}$ parallel and perpendicular to the radial direction. However, even though these integrals can be evaluated exactly, the explicit expressions are rather lengthy, because of the complex structure of the integration regions of Figs. 2 and 3. They are given in Appendix 2 together with an outline how they are obtained. As has been developed at the end of Section 1, these expressions supply the foundation for the selfconsistent

determination of the electric potential $V(r)$. The numerical procedure, which was used to calculate the solutions, and the above mentioned problem of the existence of solutions for $0 < r - r_m \ll r_0$ are discussed in Appendix 3.

3. Presentation and Discussion of the Results

As mentioned earlier, in this paper models of the solar wind are considered either with collisionless protons together with collision-dominated electrons for which an isotropic temperature is explicitly prescribed, or with two collisionless plasma components. Before the results can be presented it is necessary to specify and to justify the choice of the free parameters inherent in the models. I begin with the model with collisionless electrons and protons.

Since our knowledge on the difference of the electron and proton temperatures in the corona is scarce, it seems natural to suppose that these temperatures are equal. Then the mean free path corresponding to Coulomb collisions will differ for the two kinds of particles only by a factor $\sqrt{2}$ and we can prescribe the transition into the collisionless region for both plasma components at the same distance r_0 . Thus there remain as free parameters of the solution only the following three quantities which are the same for electrons and protons: the distance of the transition layer r_0 , the density $n(r_0)$, and the temperature T_0 of the maxwellian distribution just below r_0 . The distance r_0 should satisfy two conditions. Since it locates the inner boundary of the collisionless region, the scale height there should be just comparable to the Coulomb mean free path. This occurs, according to Biermann, at about ten solar radii. Further, because of the assumption of a static atmosphere below r_0 the flow velocity above r_0 must be small compared with the thermal velocities, which is only correct if r_0 is located near the solar surface. The compromise made in this paper is $r_0 = 2.5 r_{\odot}$. Newkirk gives in his density model of the corona for the density at a distance of $2.5 r_{\odot}$ the value $n = 9 \cdot 10^5 \text{ cm}^{-3}$, which has been adopted here for the density $n(r_0)$. The choice of the temperature is $T_0 = 1.32 \cdot 10^6 \text{ }^\circ\text{K}$. In the following I shall call this model Model I.

In the case of the isotropic electrons the location of the transition layer, which here refers only to the protons, $r_0 = 2.5 r_{\odot}$ and the parameters $n(r_0) = 9 \cdot 10^5 \text{ cm}^{-3}$, $T_0 = 1.32 \cdot 10^6 \text{ }^\circ\text{K}$ are the same as above. We consider three models in which the

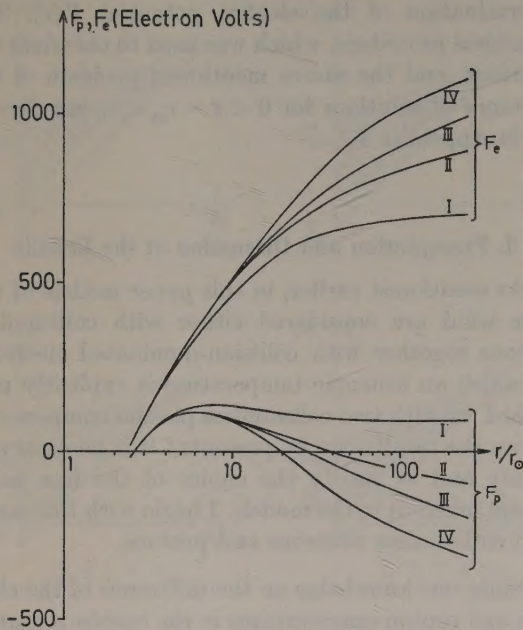


Fig. 4. The potential energies F_p and F_e of the protons and electrons as functions of the distance r from the solar center. Because of the small electron mass the potential F_e represents very nearly the negative electric potential. The electric field depends mainly on the electron temperature. Roughly speaking, if the electrons are hotter and therefore have more energy, a greater electric field is necessary to preserve charge neutrality and to transfer the thermal electron energy into energy of the proton bulk velocity. Hence, according to the electron temperature shown in Fig. 6, if we proceed from Model I to IV, F_e increases and F_p decreases

electron temperature is prescribed as follows:

$$\text{Model II: } T_e = \text{constant, for } r \leq 9 r_\odot,$$

$$T_e \propto r^{-1/2}, \quad \text{for } r > 9 r_\odot.$$

$$\text{Model III: } T_e = \text{constant, for } r \leq 9 r_\odot,$$

$$T_e \propto r^{-1/3}, \quad \text{for } r > 9 r_\odot.$$

$$\text{Model IV: } T_e = \text{constant, for } r \leq 25 r_\odot,$$

$$T_e \propto r^{-1/2}, \quad \text{for } r > 25 r_\odot.$$

The isothermal interval which originally was introduced in order to simplify the calculations, can be thought of as an extended region of nonthermal heating, as expected by Gordon, Barnes and several other authors. If we include heat conduction but neglect viscosity and the energy exchange with the protons in the region $r > 9 r_\odot$, the energy flux of the electrons of Model III, according to hydrodynamic

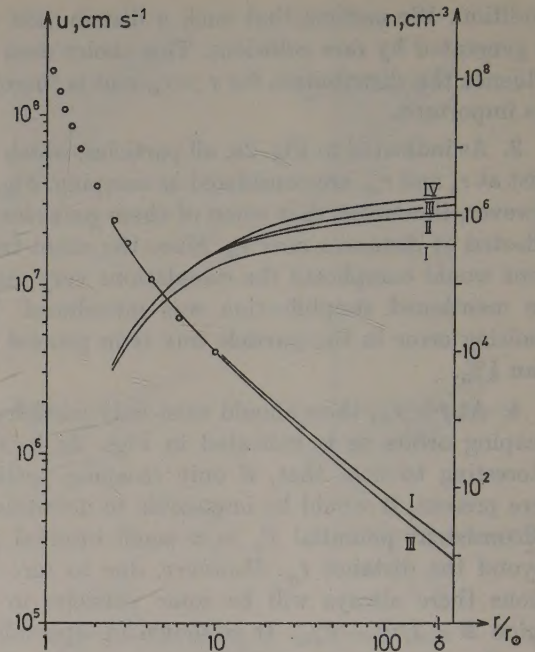


Fig. 5. The bulk velocity u and density n versus the distance r . The circles refer to Newkirk's density data of the solar corona, which fit well into the density profile of the model. Near the earth, the distance for which is marked by the symbol $\circ$, the velocity values of the Model II—IV fall within the range of the observations

theory, changes only by a factor 1.5 between $9 r_\odot$ and earth. Thus, this model can be considered approximately as one with heat conducting electrons, which are described by hydrodynamic theory. Model II, however, leads to values of the electron temperature near the earth close to those measured by Montgomery *et al.* (1968). In Model IV the electron temperature was prescribed in such a way that near the earth it equalizes the electron temperature of Model III, even though there is a much more extended isothermal region.

It should be mentioned that in contrast to the assumption of a static atmosphere just below the transition layer also the opposite approach, namely a highly supersonic flow at that level, is possible and has been used by Hollweg quite recently. He considers a model with isothermal electrons, which are treated macroscopically, and collisionless protons and locates their transition layer at distances between ten and thirty solar radii. His results agree with ours quite well. His model, however, has the disadvantage that the bulk velocity at the transition layer appears as an additional free parameter. Therefore his model

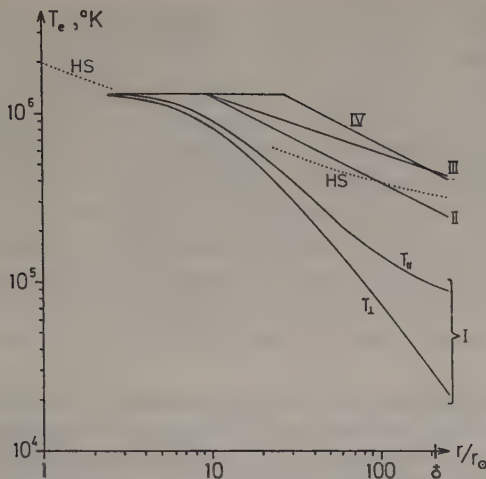


Fig. 6. The electron temperature as function of the distance r . In the Models II–IV it was explicitly prescribed, whereas in the Model I with two collisionfree components it follows from the calculations. Since the electron distribution is anisotropic we must, in the latter case, distinguish between the temperatures $T_{e||}$ and $T_{e\perp}$. Because of the plateau of the temperature $T_{e||}$ at small distances a maximum of the proton potential F_p (Fig. 4) is possible also in Model I, even though the decline of the temperature $T_{e||}$ is unrealistically rapid. The dotted line represents the isotropic electron temperature of the model of Hartle and Sturrock

is only conditionally suited to study the characteristic differences between collisionless and collision-dominated models of the solar wind.

The results are presented in the Figs. 4–8. In Table 1 some observable quantities obtained from the models are compared with the corresponding measurements and with the values from the hydrodynamic two-component model of Hartle and Sturrock (1968). The observed values are taken from Hundhausen's review article and refer to quiet sun conditions. The comparison of the Models I–IV with the observations leads to the following points.

1. The values predicted by the Model I are much more realistic than those of Chamberlain (1960), who, using the unrealistic Pannekoek-Rosseland field, calculated a flow velocity near the earth of only 20 km s^{-1} . Since our solution describes transonic flow, we see that the famous Parker-Chamberlain controversy rested on the fact that Chamberlain did not treat the electric field correctly, and does actually not exist. Nevertheless the electron temperature and hence the acceleration of the particles in Model I is somewhat too low to describe the measurements correctly.

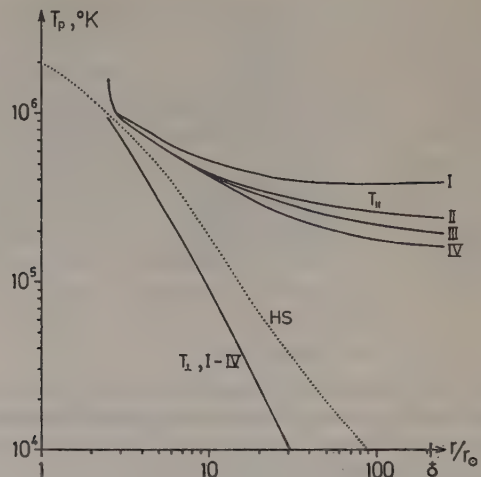


Fig. 7. The proton temperature as function of the distance r . In all four models the proton temperature $T_{p||}$ in the radial direction exceeds the values observed near the earth in the quiescent solar wind. Generally the dispersion of velocities diminishes, if the corresponding particles are accelerated. Hence in the region $r > r_m$ the temperature $T_{p||}$ decreases with the potential F_p . The temperature $T_{p\perp}$ is far too low. The dotted line represents the isotropic proton temperature of the model of Hartle and Sturrock

2. The velocity values of the Models II–IV, in which an isotropic electron temperature was explicitly prescribed, agree well with the observations. As in the hydrodynamic solar wind models the density is too high. According to Hundhausen the low observed densities may be caused by the fact, that near the sun closed magnetic field lines prevent a considerable part of the solar surface from contributing to the solar wind.

3. Near the earth the mean proton temperatures of all models exceed somewhat the observed values, while the ratio $T_{p||}/T_{p\perp}$ is very much too high. However, only about 2.5 collisions per a.u. seem to be necessary to reduce the extreme anisotropy of the collisionless case to the value observed, as was shown by Griffel and Davis. These authors in their one-component model of the solar wind include the solar rotation and treat the rare collisions with the Bhatnagar-Gross-Krook equation. Because of the collisions also the temperature is somewhat lower than in the completely collisionfree case. The fact that the magnetic lines of force actually have the form of an Archimedes spiral, according to the equations of Griffel and Davis, has a minor influence on the proton temperatures.

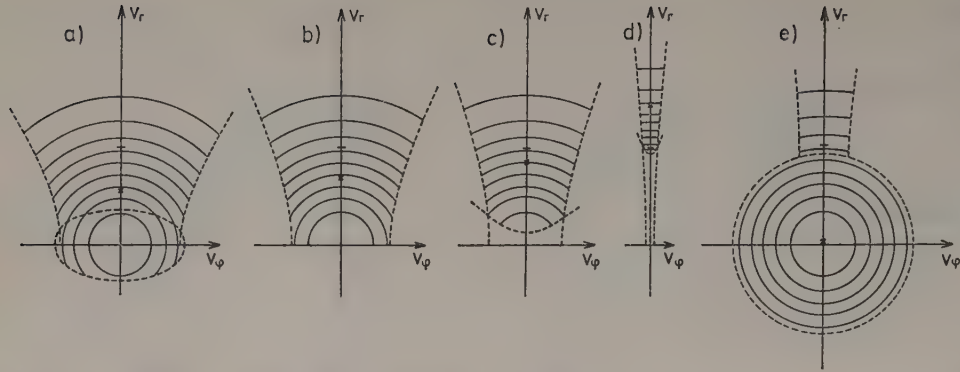


Fig. 8. The distribution functions in velocity space corresponding to Model III. Since they are cylindrically symmetric with respect to the v_r -axis, we can confine ourselves to the $\{v_r, v_\phi\}$ -plane. a)–d) are proton distributions at different distances: a) $r = 4.5 r_\odot$, b) $r = 5.5 r_\odot$, c) $r = 7 r_\odot$, d) $r = 30 r_\odot$. e) is an electron distribution at $r = 25 r_\odot$. All distribution functions are truncated Maxwellian distributions which along the dotted boundaries discontinuously decrease to zero. Lines are drawn at equidistant levels of 1/10 of the maximum value of the distribution function. The cross denotes the mean velocity. At the bar on the v_r -axis the velocity equals $(2kT_0/m_{p,e})^{1/2}$. The dashed ellipse in a) and the dashed circle in e) enclose the trapped particles.

Note the asymmetry with respect to the solar and antisolar direction, which is also observed

Table 1

Observable quantities at the earth's distance	Model I	Model II	Model III	Model IV	Hartle and Sturrock	Observations
Flux density of the protons, $10^8 \text{ particles s}^{-1} \text{ cm}^{-2}$	3.2	3.5	3.5	3.5	3.8	1.6
Density of the protons, particles cm^{-3}	18	14	12	11	15	5
Bulk velocity, km s^{-1}	172	253	288	322	250	320
Mean proton temperature $(T_{p } + 2T_{p\perp})/3$, $10^4 \text{ }^\circ\text{K}$	12.9	8.1	6.7	5.6	.44	4
Proton anisotropy $T_{p }/T_{p\perp}$	1900	1100	900	790	1	~ 2
Mean electron temperature $(T_{e } + 2T_{e\perp})/3$, $10^5 \text{ }^\circ\text{K}$	.5	2.7	4.6	4.5	3.4	~ 1
Electron anisotropy $T_{e }/T_{e\perp}$	3.5	1	1	1	1	~ 1.1

4. Apart from the anisotropy of the distribution function, which is due to the fact that $T_{||}/T_{\perp}$ is different from unity, all distribution functions of Fig. 8 have a lower velocity dispersion in the solar than in the antisolar direction. In the case of the protons this type of anisotropy is also observed (Hundhausen *et al.*, 1967) Khokhlov (1968) has pointed out that also the spiral magnetic field may lead to such an anisotropy. From the more detailed calculations by Griffel and Davis (1969) we see, however, that this effect is too small to explain the observations. The truncating of the distribution function, as it must partly be present also in the actual solar wind, may be a more effective mechanism to generate this anisotropy.

In order to determine the kinetic state of the solar wind, Hartle and Sturrock, based on the data of their model calculate the characteristic collision frequencies and compare them with the expansion frequency $u \cdot d \ln n / dr$. They find that both plasma components should be collision-dominated. In the case of the protons this is caused by Hartle and Sturrock's very low proton temperatures. If one uses the values of our Models II–IV one gets the result the kinetic situation of the protons in the direction outward the sun tends to become more and more collisionless, but the expansion frequency at the earth exceeds the proton-proton collision frequency only by an order of magnitude. Since the flow velocity of the electrons is subsonic, according to Biermann (1969) the ade-

quate macroscopic frequency is given by the reciprocal scale height $d \ln n / dr$ multiplied with the thermal electron velocity instead of the bulk velocity. Accordingly the electrons must be considered more collisionless. Using the data of the Models II–IV we find a kinetic behaviour of the electrons similar to that of the protons. Since the temperature, density, and velocity values of the Models II–IV are quite realistic it follows that in the solar wind between the sun and the earth the protons as well as the electrons are neither collision-dominated nor completely collisionless.

Since the kinetic behaviour of the protons is intermediate between that assumed in Hartle and Sturrock's model and that of our Models II–IV, also the observed values should lie between the values predicted by these two types of models. For the proton temperatures indeed this is the case, while the bulk velocities and electron temperatures in both kinds of models roughly agree with the observations. The density values, which are nearly the same in all models, in this connection cannot be compared with the observations, since they contain the systematic error caused by the magnetic channeling of the flow near the solar surface. Hence it seems possible, that

the proton temperatures and the proton distribution observed near the earth for the quiescent solar wind can be understood completely on the grounds of ordinary Coulomb collisions of the protons with each other. However, if the wind is disturbed, nonthermal heating must be present, as proposed by Barnes (1969) as well as by Biermann and Pfirsch (1969) and some other authors. For, if in the disturbed solar wind only the electrons were heated, the proton temperature should decrease according to the Models II–IV, a fact, which contradicts the observations.

Because the Coulomb collisions are not sufficient to guarantee the isotropic electron distribution, comparing the Model I with the Models II–IV we expect a bulk velocity lower than predicted by our Models II–IV. Also the electron mean temperature should be reduced to values smaller than 10^5 °K. Since this is actually not observed we must conclude, that in the solar wind wave-particle interactions and other disturbances always cause a fluidlike behaviour of the electrons.

I would like to thank Prof. Dr. L. Biermann, Dr. F. Meyer, Dr. D. Pfirsch, H. Schamel and Dr. H. U. Schmidt for their constant availability for discussions and their many helpful suggestions.

Appendix 1

A Simple Solar Wind Model with Isothermal Electrons and Purely Radially Moving Collisionfree Protons

In order to construct a very simple but instructive model of a collisionless solar wind we assume isothermal electrons and purely radially moving collisionfree protons. In this case the boundary condition for the proton distribution function for escaping orbits – characterized here by $E - F_{pm} > 0$ – has the form

$$f_p(r_0) = r_0^2 n_0 (2\pi m_p)^{-3/2} (kT_0)^{-1/2} e^{-E/kT_0} \delta(I) \frac{1 + \text{sign}(p_r)}{2}. \quad (\text{A1.1a})$$

For the trapped particles, i.e. if $0 < E < F_{pm}$, the term involving p_r must be omitted:

$$f_p(r_0) = r_0^2 n_0 (2\pi m_p)^{-3/2} (kT_0)^{-1/2} e^{-E/kT_0} \delta(I). \quad (\text{A1.1b})$$

The proton distribution function for $E - F_{pm} > 0$ is determined in the whole interval $r > r_0$ by this boundary condition to be equal to (A1.1a), whilst the distribution function has the form (A1.1b) for $0 < E < F_{pm}$ in the interval $r_0 < r < r_m$. It remains to prescribe the distribution function in two regions: For $r > r_m$ and $E < F_{pm}$, as in the general case, we define $f_p = 0$. In the small minimum near r_0 characterized by $F_p < 0$ and $r_0 < r < r_m$ we assume an equilibrium distribution equal to (A1.1b).

From this distribution function it is easy to calculate the density and the flux density. Using the transformation given below by the Eqs. (A2.4) and (A2.5), we obtain

$$j_p = n_0 \cdot r_0^2 \sqrt{\frac{kT_0}{2\pi m_p}} e^{-G_{pm}}. \quad (\text{A1.2})$$

$$n_p = \frac{n_0 \cdot r_0^2}{2r^2} e^{-G_p} [1 + \text{sign}(r_m - r) \Phi(\sqrt{G_{pm} - G_p})], \quad (\text{A1.3})$$

where $G_{pm} = F_{pm}/kT_0$ and $G_p = F_p/kT_0$ are dimensionless quantities. $\Phi(x)$ is the error function defined by

$$\Phi(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-y^2} dy. \quad (\text{A1.4})$$

As we have explained in Section 1, the essential fact now is that the proton density has a completely different behaviour inside and outside of the location of the maximum r_m . Since in the interval $r_0 < r < r_m$ trapped particles are present the density decreases with rising potential G_p roughly exponentially. (The error function being smaller than unity expresses the lack of incoming particles). If $r_m < r$, the density increases with increasing potential G_p , because in this region there are only escaping particles. If G_p is higher, these particles move more slowly which, because of the flux conservation, leads to a higher density.

The selfconsistent potential $V(r)$ can be determined as outlined at the end of Section 1. However, instead of the electric potential V we use the dimensionless potential G_p , which is connected with $V(r)$ by the equation

$$G_p(r) = L \left(1 - \frac{r_0}{r} \right) + \frac{eV(r)}{kT_0}. \quad (\text{A1.5})$$

This equation follows directly from (1.3). The dimensionless quantity L is defined by

$$L = M_\odot \cdot m_p \cdot \gamma / (r_0 \cdot kT_0). \quad (\text{A1.6})$$

Further we introduce the dimensionless flux density $\tilde{j}$ by

$$\tilde{j} = j \sqrt{\frac{2\pi m_p}{kT_0}} / (n_0 \cdot r_0^2). \quad (\text{A1.7})$$

For isothermal electrons of temperature T_0 we deduce from (1.6), neglecting terms proportional to m_e/m_p

$$\ln(n/n_0) = G_p(r) - L \left(1 - \frac{r_0}{r} \right). \quad (\text{A1.8})$$

With the aid of Eqs. (A1.2), (A1.3), (A1.7) and (A1.8) we can write the equations corresponding to (1.7) and (1.8):

$$\frac{r_0^2}{2r^2} e^{-G_p} \left[1 + \text{sign}(r_m - r) \Phi \left(\sqrt{-\ln \tilde{j} - G_p} \right) \right] = e^{-L(1-r_0/r) + G_p}, \quad (\text{A1.9})$$

$$\frac{r_0^2}{2r_m^2} \tilde{j} = \frac{1}{\tilde{j}} e^{-L(1-r_0/r_m)}. \quad (\text{A1.10})$$

Eliminating $\tilde{j}$ from Eq. (A1.9) by use of Eq. (A1.10) and taking the logarithm of (A1.9) we obtain

$$\begin{aligned} & -2G_p - \ln(2) + \ln \left[1 + \text{sign}(r_m - r) \Phi \left(\sqrt{\frac{L}{2} \left(1 - \frac{r_0}{r_m} \right) - \frac{1}{2} \ln 2 - \ln \frac{r_m}{r_0} - G_p} \right) \right] \\ & = 2 \ln(r/r_0) - L(1 - r_0/r). \end{aligned} \quad (\text{A1.11})$$

This is the desired equation which, for a given value of r_m , determines the potential $G_p(r)$. Since the l.h.s. of this equation depends only on G_p and the r.h.s. only on r we can construct the solution by the same graphical method as Parker has used to obtain solutions of the hydrodynamic solar wind equations. In Fig. 9a the four curves labelled I–IV give the values of the l.h.s. of Eq. (A1.11) as a function of G_p for four different values of r_m . For each value of G_p there are two values of the l.h.s., the greater value corresponding to $\text{sign}(r_m - r) = 1$ and the smaller to $\text{sign}(r_m - r) = -1$. In Fig. 9b the r.h.s. of Eq. (A1.11) is plotted as a function of r/r_0 for $L = 9$. For given values of r_m and G_p and for a given sign $r - r_m$ we find the values of r/r_0 solving Eq. (A1.11) by the following procedure. At first we select in Fig. 9a from the curve corresponding to the given value of r_m that point which has the abscissa value G_p and corresponds to the given sign of $r_m - r$. Then we look at those points of the curve in Fig. 9b, which have the same ordinate values as the point of Fig. 9a. Obviously the abscissa values of r/r_0 of these points solve Eq. (A1.11). The result, which we obtain in this way, is depicted in Fig. 10. The labels

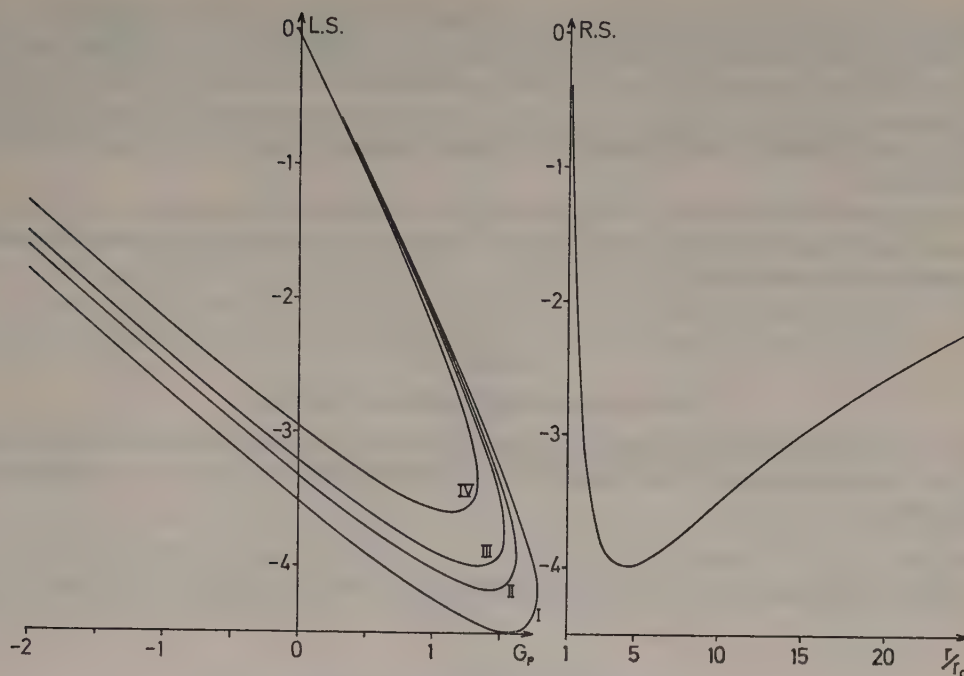


Fig. 9. Aids for the graphical solution of Eq. (A1.11)

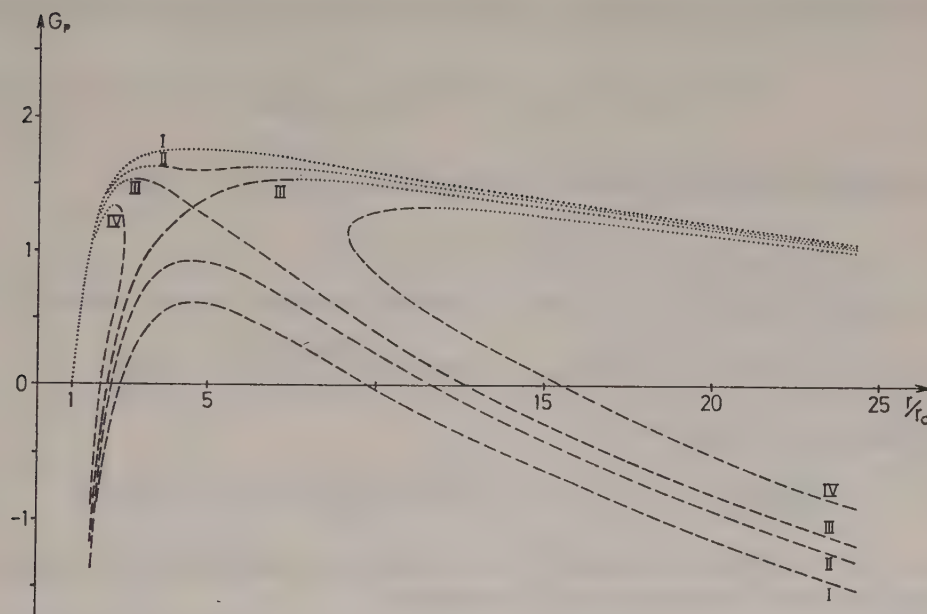


Fig. 10. The solutions of Eq. (A1.11)

I–IV refer to the four curves in Fig. 9a. The dotted lines correspond to $\text{sign}(r - r_m) = 1$ and the dashed lines to $\text{sign}(r - r_m) = -1$. It is clear that a solution consistent with (A1.11) must start at $r/r_0 = 1$ as a dotted curve, reach a maximum and then change into a dashed curve. Interestingly there is only one curve in Fig. 10 with this property, namely the “critical” curve passing through the singularity. As in Parker’s case this solution is

given by that value of r_m , for which the minima of both sides of Eq. (A1.11) coincide. Most of the neighbouring curves in Fig. 10 have no physical significance. However, in order to make the close relationship to Parker's hydrodynamic model more obvious, we note that the upper branch of solution II describes flow where outside from the sun the particle orbits are occupied in the same way as near the sun, except from a small intermediate region with no trapped particles. Thus this solution corresponds in some sense to the subsonic solution of Parker's hydrodynamic theory.

Even though the model described in this Appendix neglects completely the transverse motion of the protons it should be mentioned that it predicts velocities and temperatures of the solar wind which agree fairly well with the observations (Jockers, 1968).

Appendix 2

The Moments of the Distribution Function

As mentioned in Section 2, the flux density, the particle density and the temperatures from the distribution function can be evaluated exactly. These calculations are sketched here. At first we write down the integrals which must be solved:

$$j = \frac{1}{\cos \varphi} \int_{-\infty}^{+\infty} \frac{p_r}{m} f dp_r dp_\varphi dp_\theta, \quad n = \frac{1}{r^2 \cos \varphi} \int_{-\infty}^{+\infty} f dp_r dp_\varphi dp_\theta, \quad T_{||} = \frac{1}{knr^2 \cos \varphi} \int_{-\infty}^{+\infty} \left(\frac{p_r}{m} - u \right)^2 f dp_r dp_\varphi dp_\theta,$$

$$T_{\perp} = \frac{m}{knr^2 \cos \varphi} \int_{-\infty}^{+\infty} \frac{I}{r^2} f dp_r dp_\varphi dp_\theta. \quad (\text{A2.1})$$

Instead of the temperature $T_{||}$ we calculate the expectation value $\langle v_r^2 \rangle$ of the squared radial velocity, which is defined by

$$\langle v_r^2 \rangle = \frac{1}{nr^2 \cos \varphi} \int_{-\infty}^{+\infty} \frac{p_r^2}{m^2} f dp_r dp_\varphi dp_\theta, \quad (\text{A2.2})$$

and which is connected with the temperature by the equation

$$T_{||} = \frac{m}{k} (\langle v_r^2 \rangle - u^2). \quad (\text{A2.3})$$

If j and n are known, u follows from the relation $j = nur^2$. We introduce the new variables H, J, ψ .

$$H = E/kT_0; \quad J = I/kT_0; \quad \psi = \arctg \frac{p_\theta + eB_0 r_0^2 \sin \varphi}{p_\varphi \cos \varphi} = \arctg \frac{v_\theta}{v_\varphi}. \quad (\text{A2.4})$$

The corresponding jacobian has the form

$$\frac{\partial(p_r, p_\varphi, p_\theta)}{\partial(H, J, \psi)} = \frac{(mkT_0)^2 \cos \varphi}{p_r} = (mkT_0)^{3/2} \frac{\cos \varphi}{\sqrt{2}} \left(H - \frac{J}{r^2} - G(r) \right)^{-1/2}. \quad (\text{A2.5})$$

As in Appendix 1, $G(r) = F(r)/kT_0$. In terms of the new variables the integrands do not depend on ψ . Thus we can immediately perform the integration over this variable.

We obtain:

$$j = n_0 \sqrt{\frac{kT_0}{2\pi m}} \int e^{-H} dH dJ, \quad n = \frac{n_0}{2\sqrt{\pi} r^2} \int \frac{e^{-H}}{\sqrt{H - \frac{J}{r^2} - G(r)}} dH dJ,$$

$$\langle v_r^2 \rangle = \frac{n_0}{n} \frac{kT_0}{\sqrt{\pi m}} \frac{1}{r^2} \int e^{-H} \sqrt{H - \frac{J}{r^2} - G(r)} dH dJ, \quad T_{\perp} = \frac{n_0}{n} T_0 \frac{1}{2\sqrt{\pi} r^4} \int \frac{J e^{-H}}{\sqrt{H - \frac{J}{r^2} - G(r)}} dH dJ. \quad (\text{A2.6})$$

Since the $\{H, J\}$ -plane, apart from scale factors, is identical with the $\{E, I/r_0^2\}$ -plane, the integration regions for the integrals of (A2.6) are given by the hatched and dotted regions of Figs. 2 and 3. When we integrate over the hatched regions we must multiply the integrand by a factor of two, because $\{p_r, p_\psi, p_\theta\}$ and $\{-p_r, p_\psi, p_\theta\}$ lead to the same values of H and I .

Besides the H -axis all bordering lines of the integration region are analytically represented by equations of the type

$$H = \frac{J}{r_*^2} - G(r_*) . \quad (\text{A2.7})$$

If we let $G(0) = 0$ — until now G was only defined for $r \geq r_0$ — and if we admit for r_* not only values $\geq r_0$ but also the value $r_* = 0$ we can look also upon the H -axis as represented by an equation of the type (A2.7). Thus we can build up the integration regions of the Figs. 2 and 3 by elementary regions which are analytically described by the inequalities

$$H' < H < H'' ; \quad r_1^2(H - G(r_1)) < J < r_2^2(H - G(r_2)) , \quad (\text{A2.8})$$

where r_1 and r_2 are $\geq r_0$ and, additionally, r_1 can be zero. H' and H'' are the H -coordinates of corners of the integration region and therefore the H -coordinates of dissection points of straight lines represented by (A2.7). We introduce the abbreviation $H(r_1, r_2)$ for the H -coordinate of the dissection point of the straight lines represented by $H = (J/r_1^2) - G(r_1)$ and $H = (J/r_2^2) - G(r_2)$.

It holds

$$H(r_1, r_2) = \frac{r_1^2 G(r_1) - r_2^2 G(r_2)}{r_1^2 - r_2^2} . \quad (\text{A2.9})$$

r_1 and r_2 must be different and may be $\geq r_0$ or zero. Note, that, since the potential $G(r)$ is discontinuous near r_0 , there is a difference between $H(r_*, r_0)$ and $\lim_{\substack{r \rightarrow r_0 \\ r > r_0}} H(r_*, r)$. We denote this limit by $H(r_*, r_0 +)$.

We now perform the integration over the J -interval given in (A2.8) and obtain

$$j = n_0 \sqrt{\frac{kT_0}{2\pi m}} \sum \int_{H'}^{H''} (r_2^2 - r_1^2) (H - H(r_1, r_2)) e^{-H} dH , \quad (\text{A2.10})$$

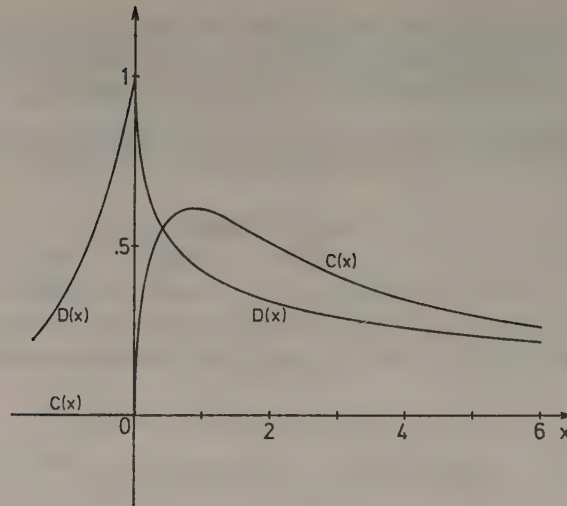
$$n = \frac{n_0}{\sqrt{\pi}} \sum \int_{H'}^{H''} \left[\sqrt{\left(1 - \frac{r_1^2}{r^2}\right) (H - H(r_1, r))} - \sqrt{\left(1 - \frac{r_2^2}{r^2}\right) (H - H(r_2, r))} \right] e^{-H} dH ,$$

$$\langle v_r^2 \rangle = \frac{n_0}{n} \frac{kT_0}{m} \frac{2}{3\sqrt{\pi}} \sum \int_{H'}^{H''} \left[\sqrt{\left(1 - \frac{r_1^2}{r^2}\right) (H - H(r_1, r))} - \sqrt{\left(1 - \frac{r_2^2}{r^2}\right) (H - H(r_2, r))} \right] e^{-H} dH ,$$

$$T_\perp = \frac{n_0}{n} T_0 \frac{1}{\sqrt{\pi}} \sum \int_{H'}^{H''} \left\{ (H - G(r)) \left[\sqrt{\left(1 - \frac{r_1^2}{r^2}\right) (H - H(r_1, r))} - \sqrt{\left(1 - \frac{r_2^2}{r^2}\right) (H - H(r_2, r))} \right] \right. \\ \left. - \frac{1}{3} \left[\sqrt{\left(1 - \frac{r_1^2}{r^2}\right) (H - H(r_1, r))} - \sqrt{\left(1 - \frac{r_2^2}{r^2}\right) (H - H(r_2, r))} \right] \right\} e^{-H} dH .$$

In (A2.10) the sum refers to the different elementary regions. The integrals lead to the complex error function. Instead of the error function we use the related functions $D(x)$ and $C(x)$:

$$D(x) = \begin{cases} e^x \cdot \frac{1}{\sqrt{\pi}} \int_x^\infty \frac{e^{-y}}{\sqrt{y}} dy & x \geq 0 , \\ e^x & x < 0 , \end{cases} \quad C(x) = \begin{cases} e^{-x} \frac{1}{\sqrt{\pi}} \int_0^x \frac{e^y}{\sqrt{y}} dy & x \geq 0 , \\ 0 & x < 0 . \end{cases} \quad (\text{A2.11})$$

Fig. 11. The functions $D(x)$ and $C(x)$

They are depicted in Fig. 11. The following formulas are useful:

$$D(x) = \begin{cases} 1 - \frac{2}{\sqrt{\pi}} \sqrt{x} + x - \frac{4}{3\sqrt{\pi}} \sqrt{x^3} + \dots & x \rightarrow 0, \quad x \geq 0, \\ \frac{1}{\sqrt{\pi x}} \left(1 - \frac{1}{2 \cdot x} + \frac{1 \cdot 3}{2 \cdot 2} \cdot \frac{1}{x^2} - \frac{1 \cdot 3 \cdot 5}{2 \cdot 2 \cdot 2} \cdot \frac{1}{x^3} + \dots \right) & x \rightarrow \infty, \end{cases} \quad (\text{A2.12})$$

$$C(x) = \begin{cases} \frac{2}{\sqrt{\pi}} \sqrt{x} - \frac{4}{3\sqrt{\pi}} \sqrt{x^3} + \dots & x \rightarrow 0, \quad x \geq 0, \\ \frac{1}{\sqrt{\pi x}} \left(1 + \frac{1}{2 \cdot x} + \frac{1 \cdot 3}{2 \cdot 2} \cdot \frac{1}{x^2} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 2 \cdot 2} \cdot \frac{1}{x^3} + \dots \right) & x \rightarrow \infty. \end{cases}$$

The integrals of (A2.10) can be evaluated with the aid of the relations

$$\begin{aligned} \frac{d}{dH} [e^{-H} (a - H - 1)] &= e^{-H} (H - a), \\ \frac{d}{dH} \left[-e^{-H} \left(\sqrt{H - a} + \frac{\sqrt{\pi}}{2} D(H - a) \right) \right] &= e^{-H} \sqrt{H - a}; \quad H > a, \\ \frac{d}{dH} \left[-e^{-H} \left(\sqrt{a - H} - \frac{\sqrt{\pi}}{2} C(a - H) \right) \right] &= e^{-H} \sqrt{a - H}; \quad H < a, \\ \frac{d}{dH} \left[e^{-H} \left(\frac{\sqrt{\pi}}{2} D(H - a) \left(b - a - \frac{3}{2} \right) + \sqrt{H - a} \left(b - H - \frac{3}{2} \right) \right) \right] &= e^{-H} \sqrt{H - a} (H - b); \quad H > a, \\ \frac{d}{dH} \left[-e^{-H} \left(\frac{\sqrt{\pi}}{2} C(a - H) \left(b - a - \frac{3}{2} \right) + \sqrt{a - H} \left(b - H - \frac{3}{2} \right) \right) \right] &= e^{-H} \sqrt{a - H} (H - b); \quad H < a. \end{aligned} \quad (\text{A2.13})$$

After some lengthy calculations we obtain for the ions ($r_m < \infty$):

Flux density;

$$j_p = \sqrt{\frac{kT_0}{2\pi m_p}} n_0 [r_m^2 e^{-G_{pm}} - (r_m^2 - r_0^2) e^{-H_p(r_0, r_m)}] \quad (\text{A2.14})$$

Particle density: (A2.15)

if $r_0 \leq r \leq r_m$:

$$n_p = n_0 \cdot \frac{1}{2} \left\{ 2 e^{-G_p} - e^{-G_{pm}} D(G_{pm} - G_p) \right. \\ \left. - \sqrt{1 - \frac{r_0^2}{r^2}} [e^{-H_p(r_0, r_m)} D(H_p(r_0, r_m) - H_p(r_0, r)) + 2e^{-H_p(r_0 +, r)} - 2e^{-H_p(r_0 +, r_m)} D(H_p(r_0 +, r_m) - H_p(r_0 +, r))] \right. \\ \left. + \sqrt{\frac{r_m^2}{r^2} - 1} [2C(H_p(r_m, r) - H_p(r_0 +, r_m)) e^{-H_p(r_0 +, r_m)} \right. \\ \left. - C(H_p(r_m, r) - G_{pm}) e^{-G_{pm}} - C(H_p(r_m, r) - H_p(r_0, r_m)) e^{-H_p(r_0, r_m)}] \right\};$$

if $r = r_m$:

$$n_p = n_0 \cdot \frac{1}{2} \left\{ e^{-G_{pm}} - \sqrt{1 - \frac{r_0^2}{r_m^2}} e^{-H_p(r_0, r_m)} \right\};$$

if $r \geq r_m$:

$$n_p = n_0 \cdot \frac{1}{2} \left\{ e^{-G_{pm}} D(G_{pm} - G_p) - \sqrt{1 - \frac{r_0^2}{r^2}} e^{-H_p(r_0, r_m)} D(H_p(r_0, r_m) - H_p(r_0, r)) \right. \\ \left. - \sqrt{1 - \frac{r_m^2}{r^2}} [e^{-G_{pm}} D(G_{pm} - H_p(r_m, r)) - e^{-H_p(r_0, r_m)} D(H_p(r_0, r_m) - H_p(r_m, r))] \right\}.$$

Temperature parallel to the radial direction: (A2.16)

if $r_0 \leq r \leq r_m$:

$$T_{p||} = T_0 \left(1 + \frac{n_0}{2n_p} \left\{ \frac{r_m^2}{r^2} \sqrt{\frac{r_m^2}{r^2} - 1} \left[C(H_p(r_m, r) - H_p(r_0, r_m)) e^{-H_p(r_0, r_m)} - 2C(H_p(r_m, r) \right. \right. \right. \\ \left. \left. - H_p(r_0 +, r_m)) e^{-H_p(r_0 +, r_m)} + C(H_p(r_m, r) - G_{pm}) e^{-G_{pm}} - \frac{2}{\sqrt{\pi}} e^{-G_{pm}} \sqrt{H_p(r_m, r) - G_{pm}} \right] \right. \\ \left. + \frac{r_0^2}{r^2} \sqrt{1 - \frac{r_0^2}{r^2}} [D(H_p(r_0, r_m) - H_p(r_0, r)) e^{-H_p(r_0, r_m)} + 2e^{-H_p(r_0 +, r)} - 2D(H_p(r_0 +, r_m) \right. \\ \left. - H_p(r_0 +, r)) e^{-H_p(r_0 +, r_m)}] + \sqrt{1 - \frac{r_0^2}{r^2}} \frac{2}{\sqrt{\pi}} \frac{r_0^2 - r_m^2}{r^2} [e^{-H_p(r_0, r_m)} \sqrt{H_p(r_0, r_m) - H_p(r_0, r)} \right. \\ \left. \left. - 2e^{-H_p(r_0 +, r_m)} \sqrt{H_p(r_0 +, r_m) - H_p(r_0 +, r)}] \right\} \right) - \frac{m_p}{k} u_p^2;$$

if $r \geq r_m$:

$$T_{p||} = T_0 \left(1 + \frac{n_0}{2n_p} \left\{ \frac{r_m^2}{r^2} \sqrt{1 - \frac{r_m^2}{r^2}} \left[D(G_{pm} - H_p(r_m, r)) e^{-G_{pm}} - D(H_p(r_0, r_m) - H_p(r_m, r)) e^{-H_p(r_0, r_m)} \right. \right. \right. \\ \left. \left. + \frac{2}{\sqrt{\pi}} \sqrt{G_{pm} - H_p(r_m, r)} e^{-G_{pm}} \right] + \frac{r_0^2}{r^2} \sqrt{1 - \frac{r_0^2}{r^2}} e^{-H_p(r_0, r_m)} \left[D(H_p(r_0, r_m) - H_p(r_0, r)) \right. \right. \\ \left. \left. + \frac{2}{\sqrt{\pi}} \sqrt{H_p(r_0, r_m) - H_p(r_0, r)} \left(1 - \frac{r_m^2}{r_0^2} \right) \right] \right\} \right) - \frac{m_p}{k} u_p^2.$$

Temperature perpendicular to the radial direction:

(A2.17)

if $r_0 \leq r \leq r_m$:

$$\begin{aligned}
 T_{p\perp} = T_0 & \left(1 + \frac{n_0}{2n_p} \left\{ \frac{r_m^2}{r^2} \sqrt{\frac{r_m^2}{r^2} - 1} \left[\left(G_{pm} - H_p(r_m, r) - \frac{1}{2} \right) (C(H_p(r_m, r) - H_p(r_0, r_m)) e^{-H_p(r_0, r_m)} \right. \right. \right. \\
 & - 2C(H_p(r_m, r) - H_p(r_0, r_m)) e^{-H_p(r_0, r_m)} + C(H_p(r_m, r) - G_{pm}) e^{-G_{pm}} \\
 & + \frac{1}{\sqrt{\pi}} \sqrt{H_p(r_m, r) - H_p(r_0, r_m)} e^{-H_p(r_0, r_m)} - \frac{2}{\sqrt{\pi}} \sqrt{H_p(r_m, r) - H_p(r_0, r_m)} e^{-H_p(r_0, r_m)} \\
 & + \frac{1}{\sqrt{\pi}} \sqrt{H_p(r_m, r) - G_{pm}} e^{-G_{pm}} \left. \left. - \frac{r_0^2}{r^2} \sqrt{1 - \frac{r_0^2}{r^2}} \left[(e^{-H_p(r_0, r)} - D(H_p(r_0, r_m) - H_p(r_0, r)) e^{-H_p(r_0, r_m)} \right. \right. \right. \\
 & \cdot 2 \cdot \left(\frac{1}{2} + H_p(r_0, r) \right) + \left(\frac{1}{2} + H_p(r_0, r) \right) \cdot D(H_p(r_0, r_m) - H_p(r_0, r)) e^{-H_p(r_0, r_m)} \\
 & \left. \left. - \frac{2}{\sqrt{\pi}} \sqrt{H_p(r_0, r_m) - H_p(r_0, r)} e^{-H_p(r_0, r_m)} + \frac{1}{\sqrt{\pi}} \sqrt{H_p(r_0, r_m) - H_p(r_0, r)} e^{-H_p(r_0, r_m)} \right] \right\} \right);
 \end{aligned}$$

if $r \geq r_m$:

$$\begin{aligned}
 T_p = T_0 & \left(1 + \frac{n_0}{2n_p} \left\{ \frac{r_m^2}{r^2} \sqrt{1 - \frac{r_m^2}{r^2}} \left[\left(G_{pm} - H_p(r_m, r) - \frac{1}{2} \right) (D(G_{pm} - H_p(r_m, r)) e^{-G_{pm}} \right. \right. \right. \\
 & - D(H_p(r_0, r_m) - H_p(r_m, r)) e^{-H_p(r_0, r_m)} + \frac{1}{\sqrt{\pi}} \sqrt{H_p(r_0, r_m) - H_p(r_m, r)} e^{-H_p(r_0, r_m)} - \frac{1}{\sqrt{\pi}} \sqrt{G_{pm} - H_p(r_m, r)} e^{-G_{pm}} \\
 & \left. \left. - \frac{r_0^2}{r^2} \sqrt{1 - \frac{r_0^2}{r^2}} \left[\left(\frac{1}{2} + H_p(r_0, r) \right) D(H_p(r_0, r_m) - H_p(r_0, r)) + \frac{1}{\sqrt{\pi}} \sqrt{H_p(r_0, r_m) - H_p(r_0, r)} e^{-H_p(r_0, r_m)} \right] \right\} \right).
 \end{aligned}$$

Since we have defined the functions D and C in (A2.11) also for $x < 0$ in the appropriate way, the arguments of these functions are allowed to become negative. However, there is one exception. In the interval $r_0 \leq r \leq r_m$ the Eqs. (A2.16) and (A2.17) are valid only, if $H_p(r_0, r_m) - H_p(r_0, r) \geq 0$. If in (A2.16) and (A2.17) appear negative radicands, the corresponding squareroots should be set equal to zero. In the case of $r = r_0$ or $r = r_m$ the expressions multiplied with $(1 - r_0^2/r^2)^{1/2}$ or with $(1 - r_m^2/r^2)^{1/2}$, respectively, vanish. Hence we need not be concerned how to define $H_p(r_0, r_0)$ or $H_p(r_m, r_m)$.

The corresponding formulas for the electrons either can be evaluated directly, or they can be deduced from the expressions for the protons if we let r_m and $-G_p(r_0 + 0)$ go to infinity. In the second case we must extensively use the formulas (A2.12). We obtain:

Flux density:

$$j_e = \sqrt{\frac{kT_0}{2\pi m_e}} n_0 r_0^2 e^{-G_{em}} (G_{em} + 1). \quad (\text{A2.18})$$

Particle density:

$$\begin{aligned}
 n_e = n_0 e^{-G_e} & \left\{ 1 - \frac{1}{2} e^{G_e - G_{em}} \cdot \left[D(G_{em} - G_e) + \frac{2}{\sqrt{\pi}} \sqrt{G_{em} - G_e} + \sqrt{1 - \frac{r_0^2}{r^2}} \left(D(G_{em} - H_e(r_0, r)) \right. \right. \right. \\
 & \left. \left. + \frac{2}{\sqrt{\pi}} \sqrt{G_{em} - H_e(r_0, r)} \right) \right] \right\}.
 \end{aligned} \quad (\text{A2.19})$$

Temperature parallel to the radial direction:

$$\begin{aligned}
 T_{e\parallel} = T_0 & \left(1 + \frac{n_0}{2n_e} e^{-G_{em}} \left\{ \frac{r_0^2}{r^2} \sqrt{1 - \frac{r_0^2}{r^2}} \left[D(G_{em} - H_e(r_0, r)) + \frac{2}{\sqrt{\pi}} \sqrt{G_{em} - H_e(r_0, r)} \right] \right. \right. \\
 & \left. \left. - \frac{2}{\sqrt{\pi}} \cdot \frac{2}{3} \left[\sqrt{G_{em} - G_e}^3 + \sqrt{\left(1 - \frac{r_0^2}{r^2} \right) (G_{em} - H_e(r_0, r))^3} \right] \right\} \right) - \frac{m_e}{k} u_e^2.
 \end{aligned} \quad (\text{A2.20})$$

Temperature perpendicular to the radial direction:

$$T_{e\perp} = T_0 \left(1 - \frac{n_0}{n_e \sqrt{\pi}} e^{-G_{em}} \cdot \left\{ \frac{2}{3} \left(\sqrt{G_{em} - G_e^3} + \sqrt{\left(1 - \frac{r_0^2}{r^2} \right) (G_{em} - H_e(r_0, r))} \right)^3 \right. \right. \\ \left. \left. + \frac{r_0^2}{r^2} \sqrt{1 - \frac{r_0^2}{r^2}} \left[\frac{\sqrt{\pi}}{2} D(G_{em} - H_e(r_0, r)) \left(\frac{1}{2} + H_e(r_0, r) \right) + \sqrt{G_{em} - H_e(r_0, r)} \left(\frac{1}{2} + G_{em} \right) \right] \right\} \right). \quad (\text{A2.21})$$

The formulas (A2.20) and (A2.21) are valid only if $G_{em} - H_e(r_0, r) \geq 0$.

Appendix 3

The Determination of the Selfconsistent Electric Potential

This appendix consists of two parts. In the first part some details about the numerical determination of the selfconsistent electric potential are given. The second part deals with the problem mentioned on page 17, point 4, that in the interval $0 < r - r_m \ll r_0$ selfconsistent solutions without trapped particles do not exist.

From the equations for the density and flux derived in Appendix 2 it is clear that the electric potential can only be determined numerically. However, the numerical calculations are simple, because we only need to search for solutions of algebraic equations. The calculations follow the general outline of Section 1, but instead of the quantities $V(r)$ and j , it is convenient to use $G_p(r)$ and G_{pm} , from which $V(r)$ and j can be determined by (A1.5) and (A2.14). Since the Eqs. (A2.18) – (A2.21) contain the electron potential G_e the relation connecting G_e and G_p

$$G_p + G_e = L \left(1 + \frac{m_e}{m_p} \right) \left(1 - \frac{r_0}{r} \right), \quad (\text{A3.1})$$

which follows from (1.3), is useful. The validity of (A3.1) rests upon the fact that T_0 is the same for electrons and protons (cf. Section 3). We perform the calculations in four steps:

1. From a given value r_m we determine G_{pm} and, if the electrons are collisionless, G_{em} . In the case of macroscopically described electrons we start from Eq. (1.6). If we assume $T_e \equiv T_0$ and use the conditions $n_e = n_p$, $j_e = j_p$ and the Eqs. (A2.14), (A2.15) we obtain for $r = r_m$:

$$2G_{pm} = L \left(1 + \frac{m_e}{m_p} \right) \left(1 - \frac{r_0}{r_m} \right) - \ln(2) + \ln \left[1 - \sqrt{1 - \frac{r_0^2}{r_m^2}} \exp \left(- \frac{r_0^2 G_{pm}}{r_m^2 - r_0^2} \right) \right] \\ + \frac{m_e}{4\pi m_p} \left[\frac{r_m^2}{r_0^2} e^{-G_{pm}} - \left(\frac{r_m^2}{r_0^2} - 1 \right) e^{-H_p(r_0, r_m)} \right]^2 \cdot \left[\frac{4r_0^4}{r_m^4} \left(e^{-G_{pm}} - \sqrt{1 - \frac{r_0^2}{r_m^2}} e^{-H_p(r_0, r_m)} \right)^{-2} - 1 \right]. \quad (\text{A3.2})$$

This equation corresponds to (1.8). For a given value of r_m from this Eq. G_{pm} is determined numerically by iteration starting from the initial value $2G_{pm} = L(1 + m_e/m_p)(1 - r_0/r_m) - \ln(2)$. In the case of collisionless electrons we use Eqs. (A2.14), (A2.15), (A2.18), (A2.19). If we insert them into the conditions $n_e = n_p$, $j_e = j_p$ we obtain the following two equations:

$$2G_{pm} = L \left(1 + \frac{m_e}{m_p} \right) \left(1 - \frac{r_0}{r_m} \right) - \ln(2) + \ln \left[1 - \sqrt{1 - \frac{r_0^2}{r_m^2}} \exp \left(- \frac{r_0^2 G_{pm}}{r_m^2 - r_0^2} \right) \right] \\ - \ln \left\{ 1 - \frac{1}{2} e^{G_e - G_{em}} \left[D(G_{em} - G_e) + \frac{2}{\sqrt{\pi}} \sqrt{G_{em} - G_e} \right. \right. \\ \left. \left. + \sqrt{1 - \frac{r_0^2}{r_m^2}} \left(D(G_{em} - H_e(r_0, r_m)) + \frac{2}{\sqrt{\pi}} \sqrt{G_{em} - H_e(r_0, r_m)} \right) \right] \right\}. \quad (\text{A3.3}) \\ G_{em} = \frac{1}{2} \ln \frac{m_p}{m_e} + G_{pm} - \ln \left[\frac{r_m^2}{r_0^2} - \left(\frac{r_m^2}{r_0^2} - 1 \right) \exp \left(- \frac{r_0^2 G_{pm}}{r_m^2 - r_0^2} \right) \right] + \ln(G_{em} + 1).$$

From (A3.1) follows for $r = r_m$

$$G_e = L \left(1 + \frac{m_e}{m_p} \right) \left(1 - \frac{r_0}{r_m} \right) - G_{pm}. \quad (\text{A3.4})$$

With $2G_{pm} = L(1 + m_e/m_p)(1 - r_0/r_m) - \ln 2$ as a first approximation we determine from the second equation of (A3.3) the corresponding value G_{em} by iteration, starting with $G_{em} = 0$. From Eq. (A3.4) we obtain the potential G_e . Then we evaluate the r.h.s. of the first Eq. (A3.3) and obtain an improved value of G_{pm} . Repeating this procedure we determine G_{pm} and G_{em} .

2. We search for the critical solution and determine r_m . In the case of collisionfree electrons we insert into the condition $n_e = n_p$ the Eqs. (A2.15) and (A2.19). Since we know G_{pm} and G_{em} , we can, from this equation, calculate G_p for $r > r_m$. In the case of macroscopically described electrons we use Eq. (1.6) with $T_e \equiv T_0$. If we replace n_e by n_p and j_e by j_p and insert Eqs. (A2.15) and (A2.16), we arrive again at an equation from which $G_p(r)$ can be calculated in the interval $r > r_m$. In both cases the solution $G_p(r)$ will in general not show the desired "critical" behaviour. Thus we must change r_m and repeat the procedure until the critical solution is found. The calculations were performed with Newton's method. Near the critical point, however, where two zeros lie very close together, it was necessary to apply the regula falsi.

3. When r_m is found the only unknown quantity is $G_p(r_0 +)$. We obtain it by solving the equations mentioned in point 2 at $r = r_0 +$. From the same equations we can then calculate as many values $G_p(r)$ as desired.

4. If, in the case of hydrodynamically described electrons, we want to prescribe a new temperature $T_e(r)$ in the region $r > r_c > r_m$ we proceed in the following way. We approximate the temperature function $T_e(r)$ by a step function having discontinuities at some distances r_ν ($\nu = 1, 2, 3 \dots$) and the values T_ν in the intervals $r_\nu < r \leq r_{\nu+1}$ ($\nu = 0, 1, 2 \dots; r_0, T_0$ as defined, $r_1 > r_c$). Then the integral in (1.6) becomes

$$\int_{r_0}^r \frac{1}{n_e} \frac{d}{dr} (k T_e n_e) dr = \sum_{\nu=0}^{n-1} \left(k T_\nu \ln \frac{n_p(r_{\nu+1})}{n_p(r_\nu)} \right) + k T_n \ln \frac{n_p(r_{n+1})}{n_p(r_n)}. \quad (\text{A3.5})$$

If we know $G_p(r_\nu)$ for $\nu = 1 \dots n$, we can calculate the sum easily. Inserting for $n_p(r_{n+1})$ Eq. (A2.15) we determine $G_p(r_{n+1})$ as in the isothermal case.

In Section 2 was mentioned that a selfconsistent determination of the potential G_p is not possible in a small region just beyond r_m . This will become clear, if we try to expand $G_{pm} - G_p$ in this region as a power series of $r^2/r_m^2 - 1$. Since in the equation for the proton density singular terms proportional to $(r^2/r_m^2 - 1)^{1/2}$ are present, we must use semi-integer powers. Hence we make the ansatz

$$G_{pm} - G_p = g_2 \left(\frac{r^2}{r_m^2} - 1 \right) + g_3 \left(\frac{r^2}{r_m^2} - 1 \right)^{3/2} + \dots, \quad (\text{A3.6})$$

g_2 must be nonnegative. We obtain

$$\begin{aligned} n_p &= n_{p0} + n_{p1} \left(\frac{r^2}{r_m^2} - 1 \right)^{1/2} + n_{p2} \left(\frac{r^2}{r_m^2} - 1 \right) + n_{p3} \left(\frac{r^2}{r_m^2} - 1 \right)^{3/2} + \dots, \\ n_e &= n_{e0} + n_{e2} \left(\frac{r^2}{r_m^2} - 1 \right) + n_{e3} \left(\frac{r^2}{r_m^2} - 1 \right)^{3/2} + \dots. \end{aligned} \quad (\text{A3.7})$$

From (A2.15) we deduce with the aid of (A2.12)

$$n_{p1} = \frac{n_0}{2} e^{-G_{pm} + g_2} [h(H_p(r_0, r_m) - G_{pm} + g_2) - h(g_2)]. \quad (\text{A3.8})$$

$h(x)$ is an auxiliary function defined for $x \geq 0$ by

$$h(x) = e^{-x} (D(x) + \frac{2}{\sqrt{\pi}} \sqrt{x}). \quad (\text{A3.9})$$

If we differentiate (A3.9), we see that $h(x)$ is monotonically decreasing. Hence, since $H_p(r_0, r_m) - G_{pm} \geq 0$, n_{p1} is always negative, no matter how we may choose g_2 . However, in order to make a solution of the condition $n_p = n_e$ possible, n_{p1} must vanish. Therefore, if we use the formula (A2.15) for the density, the neutrality condition cannot be satisfied for r a little bit greater than r_m .

The fact $n_{p1} < 0$ says that, compared with n_e , the proton density near r_m for $r > r_m$ declines too rapidly. In order to avoid this it is necessary to assume trapped particles also in the region $r > r_m$. Also from physical

grounds it seems very improbable that in this region trapped particles should be completely absent. Since in the region $E - (I/r_m^2) - F_{pm} < 0$ we are free to adjust a function of the two variables E and I it appears that there are many possible choices of this function leading to a coefficient n_{p1} which vanishes at some finite value of g_2 . An example, which perhaps is somewhat unrealistic but has the advantage of being tractable within the framework of our theory, is the following. We prescribe in the region $E - (I/\alpha^2 r_m^2) - F_{pm} > 0$ and $E - (I/r_m^2) - F_{pm} < 0$ a distribution of trapped particles as in (2.7b) times a factor β . Then we must add to the particle density given in (A2.15) for $r > r_m$ the following summand:

$$\beta \cdot e^{-G_{pm}} \left[\sqrt{1 - \frac{r_m^2}{r^2}} D(G_{pm} - H_p(r_m, r)) + \sqrt{\frac{\alpha^2 r_m^2}{r^2} - 1} C\left(\frac{r^2}{\alpha^2 r_m^2 - r^2} (G_{pm} - G_p)\right) \right], \quad (\text{A3.10})$$

n_{p1} becomes

$$n_{p1} = \frac{n_0}{2} e^{-G_{pm} + g_2} [h(H_p(r_0, r_m) - G_{pm} + g_2) - h(g_2) (1 - 2\beta)], \quad (\text{A3.11})$$

Now in a wide range of β values of g_2 exist which let n_{p1} vanish. In the numerical determination of the potential G_p we used for $r > r_m$ the proton density modified by (A3.10) with $\alpha = 1.05$, $\beta = 0.05$. The numerical calculations show the existence of solutions for all values of r . Since the distribution of trapped particles was chosen for practical reasons and is somewhat unrealistic, in the calculations of the temperatures and in Fig. 8 the presence of trapped particles in the interval $r > r_m$ has been neglected. This may be justified by the fact that the selfconsistent potential which in the interval $r_c \leq r < \infty$ exists also for the unmodified proton density, differs from the potential calculated by the modified proton density only by some percent.

References

- Aamont, R.E., Case, C.M. 1962, *Phys. Fluids* **5**, 1019.
 Barnes, A. 1969, *Ap. J.* **155**, 311.
 Biermann, L. 1969, Lecture Notes of the University of Colorado.
 Biermann, L., Pfirsch, D. 1969, Preprint from the Max-Planck-Institute of Astrophysics, München Nr. 16.
 Brandt, J.C., Cassinelli, J.P. 1966, *Icarus* **5**, 47.
 Chamberlain, J.W. 1960, *Ap. J.* **131**, 47.
 Dessler, A.J. 1967, *Rev. Geophys.* **5**, 1.
 Gordon, I.M. 1969, *Ap. Letters* **3**, 181.
 Griffel, D.H., Davis, L. 1969, *Pl. Sp. Sci.* **17**, 1009.
 Hartle, R.E., Sturrock, P.A. 1968, *Ap. J.* **151**, 1155.
 Hollweg, J.V. 1969, submitted to *J. Geophys. Res.*
 Hundhausen, A.J. 1968, *Sp. Sci. Rev.* **8**, 690.
 Hundhausen, A.J., Asbridge, J.R., Bame, S.J., Gilbert, H.E., Strong, I.B. 1967, *J. Geophys. Res.* **72**, 87.
 Jeans, J.H. 1954, *The Dynamical Theory of Gases*, Dover Publ. London.
 Jensen, E. 1963, *Astrophys. Norvegica* **8**, 99.
 Jockers, K. 1968, *Mitt. Astron. Ges.* **25**, 217.
 Khokhlov, M.Z. 1968, *Geomagnetizm i Aeronomija* **8**, 330.
 Lüst, R. 1962, *Sp. Sci. Rev.* **1**, 522.
 Newkirk, G. 1967, *Ann. Rev. Astron. Astrophys.* **5**, 213.
 Montgomery, M.D., Bame, S.J., Hundhausen, A.J. 1968, *J. Geophys. Res.* **73**, 4.
 Parker, E.N. 1958, *Ap. J.* **128**, 664.
 Pikel'ner, S.B. 1950, *Izv. Krymskoj Astrofiz. Obs.* **5**, 34.
 Schlüter, A. 1950, *Z. Naturforschung* **5a**, 72.
 Spitzer, L. 1962, *Physics of Fully Ionized Gases*, Interscience Publ., London.

K. Jockers
 Max-Planck-Institut für Physik und Astrophysik
 BRD-8000 München 23, Föhringer Ring 6,
 Germany

Printed in Germany by Brühlsche Universitätsdruckerei Gießen

Lebenslauf

Ich, Klaus Jockers, bin am 10.8.1940 als Sohn des Studienrates Walter Jockers und seiner Frau Helene in Regensburg geboren. Von 1946 bis 1950 besuchte ich die Volksschule in Regensburg-Prüfening und von 1950 bis 1955 das Alte Gymnasium in Regensburg. Ich bestand das Abitur im Juli 1959 am Humanistischen Gymnasium in Aschaffenburg, wohin meine Eltern 1955 übergesiedelt waren. Seit dem Wintersemester 1959/60 studiere ich Physik an der Universität München. Mein Vordiplom legte ich Februar 1962 ab, das Hauptdiplom folgte im Juli 1966. Die vorliegende Dissertation entstand in der Zeit von Oktober 1966 bis Oktober 1969 am Max-Planck-Institut für Physik und Astrophysik in München. Seit Juni 1966 bin ich verheiratet.

